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Trapped modes of internal waves in a channel spanned by a submerged cylinder

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A horizontal channel of infinite length and depth and of constant width contains inviscid, incompressible, two-layer fluid under gravity. The upper layer has constant finite depth and is occupied by a fluid of constant density ρ^*. The lower layer has infinite depth and is occupied by a fluid of constant density $\rho > \rho^*$. The parameter $\epsilon = (\rho/\rho^*) - 1$ is assumed to be small. The lower fluid is bounded internally by an immersed horizontal cylinder which extends right across the channel and has its generators normal to the sidewalls. The free, time-harmonic oscillations of fluid, which have finite kinetic and potential energy (such oscillations are called trapped modes), are investigated. Trapped modes in homogeneous fluid above submerged cylinders and other obstacles are well known. In the present paper it is shown that there are two sets of frequencies of trapped modes for the two-layer fluid. The frequencies of the first finite set are close to the frequencies of trapped modes in the homogeneous fluid (when $\rho^* = \rho$). They correspond to the trapped modes of waves on the free surface of the upper fluid. The frequencies of the second finite set are proportional to ϵ, and hence, are small. These latter frequencies correspond to the trapped modes of internal waves on the interface between two fluids. To obtain these results the perturbation method for a quadratic operator family was applied. The quadratic operator family with bounded, symmetric, linear, integral operators in the space $L_2(-\infty, +\infty)$ arises as a result of two reductions of the original problem. The first reduction allows to consider the potential in the lower fluid only. The second reduction is the same as used by Ursell (1987).

1. Introduction

Trapped modes of waves on the free surface of a homogeneous fluid are well known. By trapped modes we mean the free, time-harmonic oscillations of the fluid decaying to zero at infinity. In the case of modes trapped in a channel, waves have finite kinetic and potential energy. The first explicit solution for trapped modes of, so-called, edge waves was discovered by Stokes (1846). He showed that a wave may progress over a sloping beach along a straight coastline with the motion decaying exponentially in the offshore direction. The existence of trapped modes in a channel above a submerged horizontal cylinder spanning the sidewalls was first established by Ursell (1951). He proved that there is a finite set of frequencies of trapped waves below a certain cutoff frequency, if a circular cylinder of a sufficiently small radius is immersed in deep water. Soon after that, Jones (1953) generalized Ursell's result to cylinders of arbitrary but symmetric cross-section, and finite water depth. In 1987 Ursell gave the new proof of these theorems. Extensive numerical results were obtained by McIver & Evans (1985) for trapped modes above a circular submerged cylinder. They found that only a single mode exists for a depth of submergence of the cylinder greater than about 1.07 of its radius. As the depth of submergence is decreased, further trapped modes appear. Martin (1989) also investigated trapped modes numerically.

Trapped modes also exist if there is a crest on the channel bottom (see Jones 1953; Bonnet & Joly 1990 and references in the last paper). However, in the case of a hole in the channel bottom trapped modes do not exist as was noticed by Vainberg & Maz'ya (1973). McIver (1991) proved that there are no trapped waves if the cylinder is semi-immersed and satisfies John's condition. This result can be improved using the version of Maz'ya's identity for surface-piercing bodies suggested by Kuznetsov (1991). Callan (1990) and Callan, Linton & Evans (1991) demonstrated that trapped modes exist in the presence of some other submerged obstacles.

One can see that the problem of trapped modes in homogeneous fluid has been investigated rather well. At the same time the author knows of no paper treating the trapped modes problem for a two-layer fluid. The importance of such investigations can be recognized from the following remark by Friis, Grue & Palm (1991): 'Long underwater tube bridges [...] are proposed to be constructed across Norwegian fjords and straits'. It is well known that these fjords often are occupied by two layers of fluid. The upper layer contains fresh water the lower layer contains salt water which has a density slightly greater than the fresh water.

Now, the contents of the paper will be briefly summarized. In §2 the statement of the problem is given. Section 3 is devoted to the reduction of the original problem to one which has only one unknown potential, which describes motion in the lower fluid. This is performed with the help of the Fourier transform. The next section, 4, contains the second step of the reduction. Using the method developed by Ursell (1987) and based on the special Green function we get a family of operators involving the eigenparameter quadratically. The operators involved are bounded, symmetric, integral operators in the space $L_2(-\infty, +\infty)$. Their properties are investigated in the Appendix. There is a small parameter in the family, since it is assumed that the difference between the densities of the lower and upper fluids is small. Then it is reasonable to apply a perturbation method. This is all the more convenient because the unperturbed square operator pencil is degenerate and one of its eigenvalues is equal to zero. The perturbation procedure is developed in §5. As a result we find, under some restrictions, that the quadratic operator family has two finite sets of positive eigenvalues. There is a simple illustration which makes the results of §5 very clear. Let us consider the quadratic equation

$$(a_0+\epsilon a)\,x^2-b(1+\epsilon)\,x+\epsilon = 0,$$

where $a_0, a, b > 0$ and ϵ is a small positive parameter. The limit equation (as $\epsilon \rightarrow 0$)

$$a_0 x^2-bx = x(a_0 x-b) = 0$$

has two roots

$$x_0^{(0)} = 0, \quad x_+^{(0)} = b/a_0.$$

The expansions of the roots of the perturbed equation are

$$x_0 = \frac{1}{b}\epsilon+O(\epsilon^2), \quad x_+ = \frac{b}{a_0}+\frac{b^2(a_0-a)-a_0}{a_0^2\, b}\epsilon+O(\epsilon^2).$$

Then these roots are positive, if ϵ is sufficiently small.

The results of §5 are essentially similar to this example. The perturbed quadratic family has two finite sets of positive eigenvalues $\{\nu^{(+)}\}$ and $\{\nu^{(0)}\}$. The first set is generated by positive eigenvalues of the unperturbed pencil. The second set is generated by the zero eigenvalue of the unperturbed pencil. Section 6 contains a discussion of the hydrodynamical meaning of these eigenvalues. The set $\{\nu^{(+)}\}$ gives the frequencies of trapped modes of waves on the free surface of the upper layer. These frequencies are close to the frequencies of trapped modes of waves on the surface of homogeneous fluid without an interface. The frequencies of the second set $\{\nu^{(0)}\}$ are proportional to the small parameter and correspond to the trapped modes of internal waves on the interface between the two layers. There is a rough but informative estimate of a quotient of frequencies of trapped modes of internal and surface waves in §6.

A rigorous mathematical justification of the perturbation procedure from §5 can be obtained by means of some results in general perturbation theory for linear operators in Hilbert space (see, e.g. Friedrichs 1965; Kato 1966). This theorem will be published in another paper.

2. Statement of the problem

We consider a channel of infinite (for simplicity) depth wave vertical sidewalls. It is occupied by two-layer fluid. Fresh water with density ρ^* occupies the upper layer whose depth (without loss of generality) can be assumed to be equal to one.

We choose (x, y, z)-coordinates with the y-axis directed upwards and with the (x, z)-plane coinciding with the undisturbed interface between the two layers (see figure 1). The sidewalls lie in the planes $\{z = \pm b\}$.

Salt water with density $\rho > \rho^*$ occupies the lower part of the channel and contains a cylinder spanning the sidewalls. It has constant cross-section D, and its generators are parallel to the z-axis (see figure 1).

We shall use the linear theory of surface waves and we assume that $\epsilon = \rho/\rho^* - 1$ is a small parameter. We denote by $\phi(x, y, z, t)\,(\phi^*(x, y, z, t))$ the time-dependent velocity potential for the lower salt (upper fresh) water. These potentials must satisfy the following relations:

$$\left.\begin{array}{ll} \nabla^2\phi^* = 0 \quad \text{in} \quad W^*, & \nabla^2\phi = 0 \quad \text{in} \quad W, \\ \phi^*_z = 0 \quad \text{when} \quad z = \pm b, & \phi_z = 0 \quad \text{when} \quad z = \pm b, \\ \phi^*_{tt} + g\phi^*_y = 0 \quad \text{when} \quad y = 1, & \partial\phi/\partial n = 0 \quad \text{on} \quad S; \end{array}\right\} \tag{1}$$

$$\left.\begin{array}{l} \phi^*_y = \phi_y \quad \text{when} \quad y = 0, \\ \rho(\phi_{tt} + g\phi_y) = \rho^*(\phi^*_{tt} + g\phi^*_y) \quad \text{when} \quad y = 0. \end{array}\right\} \tag{2}$$

Here g is the acceleration due to gravity and n is the unit normal to the cylinder surface S directed into W. Relations (1) and (2) are the usual ones in the linear surface wave theory. We mention only that the conditions (2) describe the continuity of the normal component of the velocity field and of the pressure across the interface.

Solutions of (1), (2) corresponding to waves of radian frequency ω and of wavenumber k along the z-axis have the form

$$\left.\begin{array}{l} \phi(x, y, z, t) = \exp(-\mathrm{i}\omega t)\, u(x, y) \cos kz, \\ \phi^*(x, y, z, t) = \exp(-\mathrm{i}\omega t)\, u^*(x, y) \cos kz. \end{array}\right\} \tag{3}$$

To satisfy the boundary conditions on the sidewalls we have to put

$$k = \pi n/b, \quad n = 1, 2, 3, \ldots.$$

We can take $\sin kz$ instead of $\cos kz$ in (3). In what follows we suppose k to be prescribed, but its value is an arbitrary positive number.

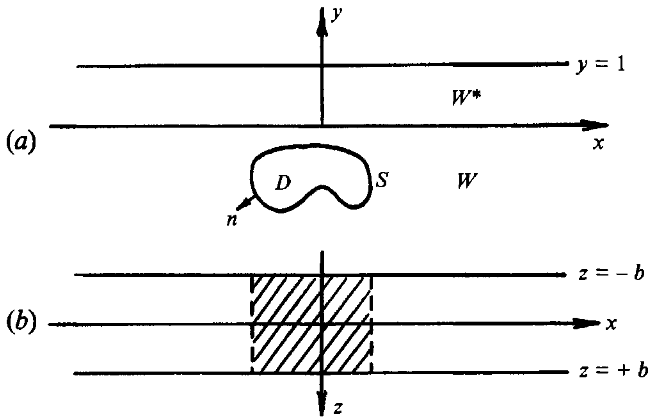

FIGURE 1. (a) Cross-section of the channel; (b) view from above.

Substituting (3) into (1) and (2), we obtain

$$\left.\begin{array}{llll} u^*_{xx}+u^*_{yy} = k^2u^* & \text{in} \quad W^*, & u_{xx}+u_{yy} = k^2u & \text{in} \quad W, \\ u^*_y - \nu u^* = 0 & \text{when} \quad y = 1, & \partial u/\partial n = 0 & \text{on} \quad S; \end{array}\right\} \tag{4}$$

$$u^*_y = u_y \quad \text{when} \quad y = 0, \tag{5a}$$

$$\rho^*(u^*_y - \nu u^*) = \rho(u_y - \nu u) \quad \text{when} \quad y = 0, \tag{5b}$$

where $\nu = \omega^2/g$.

For trapped-mode solutions the motion must decay at large distances:

$$u^*, |\nabla u^*| \to 0 \quad \text{as} \quad |x| \to \infty,$$

$$u, |\nabla u| \to 0 \quad \text{as} \quad |x+\mathrm{i}y| \to \infty.$$

More precisely, the kinetic and potential energy of surface and internal waves should be finite.

$$\left.\begin{array}{l} \displaystyle\int_{-\infty}^{+\infty} \{[u^*(x,0)]^2 + [u^*(x,1)]^2\}\,\mathrm{d}x + \int_{W^*} |\nabla u^*|^2\,\mathrm{d}x\,\mathrm{d}y < \infty, \\ \displaystyle\int_{-\infty}^{+\infty} u^2(x,0)\,\mathrm{d}x + \int_{W} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y < \infty. \end{array}\right\} \tag{6}$$

Thus, we have the homogeneous boundary value problem with the spectral parameter ν, which enters into the conditions on the free surface of upper fluid and on the interface. The last condition contains both the unknown functions u and u^*. Our aim is to reduce the problem with two unknown functions to a problem which can be treated by the usual tools of the spectral theory of linear operators (see Kato. 1966, ch. 4). The above-mentioned condition on the interface will play the essential role in the reduction, the first step of which will be made in the next section.

3. Reduction to the problem in the lower layer

Let us reduce the problem (4)–(6) with the unknown pair (u^*, u) to a problem with one unknown function u. We have to determine u^* assuming that $u_y(x,0)$ is given in (5a).

Using the Fourier transform

$$\tilde{u}^*(\xi, y) = \int_{-\infty}^{+\infty} u^*(x, y)\, e^{-ix\xi}\, dx$$

we get

$$\tilde{u}^*_{yy} = (k^2+\xi^2)\tilde{u}^*, \quad 0 < y < 1, \tag{7}$$

$$\tilde{u}^*_y - \nu\tilde{u}^* = 0, \quad y = 1, \tag{8}$$

$$\tilde{u}^*_y = \tilde{u}_y, \quad y = 0. \tag{9}$$

The general solution of (7) is

$$\tilde{u}^*(\xi, y) = C_1(\xi)\cosh\lambda y + C_2(\xi)\sinh\lambda y, \quad \lambda = (k^2+\xi^2)^{\frac{1}{2}}.$$

Conditions (8) and (9) yield the system

$$(\lambda\sinh\lambda - \nu\cosh\lambda)\, C_1 - (\nu\sinh\lambda - \lambda\cosh\lambda)\, C_2 = 0,$$

$$\lambda C_2 = \tilde{u}_y.$$

Hence,

$$C_1(\xi) = \frac{\tilde{u}_y}{\lambda}\frac{\nu\sinh\lambda - \lambda\cosh\lambda}{\lambda\sinh\lambda - \nu\cosh\lambda}, \quad C_2(\xi) = \frac{\tilde{u}_y}{\lambda}$$

and

$$\tilde{u}^*(\xi, y) = \frac{\tilde{u}_y(\xi, 0)}{\lambda}\frac{\nu\sinh\lambda(1-y) - \lambda\cosh\lambda(1-y)}{\lambda\sinh\lambda - \nu\cosh\lambda}. \tag{10}$$

Applying the Fourier transform to (5*b*) and taking into account (5*a*), we obtain

$$\epsilon\tilde{u}_y = \nu[(1+\epsilon)\tilde{u} - \tilde{u}^*], \quad y = 0.$$

We substitute (10) into the last equality and arrive at

$$\tilde{u}_y = \frac{(1+\epsilon)\nu\lambda(\lambda\tanh\lambda - \nu)}{(\nu^2+\epsilon\lambda^2)\tanh\lambda - (1+\epsilon)\nu\lambda}\tilde{u}, \quad y = 0.$$

Thus, the problem for u includes

$$u_{xx} + u_{yy} = k^2 u \quad \text{in} \quad W, \tag{11}$$

$$\partial u/\partial n = 0 \quad \text{on} \quad S, \tag{12}$$

$$u_y = \frac{(1+\epsilon)\nu}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\frac{\lambda(\lambda\tanh\lambda - \nu)}{(\nu^2+\epsilon\lambda^2)\tanh\lambda - (1+\epsilon)\nu\lambda}\tilde{u}\, d\xi \quad \text{when} \quad y = 0. \tag{13}$$

The trapped-mode solution must also satisfy (6). In (13) we have a pseudo-differential operator depending on the spectral parameter ν, and parameters k, ϵ which we consider to be prescribed. Moreover, ϵ is a small parameter.

Note that the problem (11)–(13), (6) is equivalent to the original problem (4)–(6).

4. Reduction to the spectral problem for a quadratic operator family

Following the method developed by Ursell (1987) we use the Green function $g(x, y; \sigma, 0)$ which satisfies the modified Helmholtz equation (11) with the boundary conditions (12) and

$$g_y = 0 \quad \text{when} \quad y = 0 \quad \text{and} \quad x \neq \sigma, \tag{14}$$

$$g \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{15}$$

Furthermore, the function g has a source singularity at $(\sigma, 0)$, i.e.

$$\sup\{g(x, y; \sigma, 0) - K_0(k[(x-\sigma)^2 + y^2]^{\frac{1}{2}})\} < \infty. \tag{16}$$

Here K_0 is the Macdonald function which has the representation

$$K_0(z) = \int_0^\infty \exp(-z\cosh\mu)\,\mathrm{d}\mu$$

and the behaviour

$$K_0(z) \sim \mathrm{e}^{-z}\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \quad \text{as} \quad z \to \infty,$$

$$K_0(z) \sim -\log z \quad \text{as} \quad z \to 0.$$

Ursell (1987) proved that g exists, and it can be obtained in the form

$$g(x,y;\sigma,0) = K_0(k[(x-\sigma)^2+y^2]^{\frac{1}{2}}) + \int_S m(s,\sigma)\{K_0(k[(x-X(s))^2 + (y-Y(s))^2]^{\frac{1}{2}}) + K_0(k[(x-X(s))^2+(y+Y(s))^2]^{\frac{1}{2}})\}\,\mathrm{d}s.$$

Here s is the arclength along S, while the points of S have coordinates $X(s)$, $Y(s)$. The function $m(s,\sigma)$ is the unique solution of the Fredholm equation of the second kind

$$-\pi m(s',\sigma) + \int_S m(s,\sigma)\frac{\partial}{\partial n(s')}\{K_0(k[(X(s')-X(s))^2+(Y(s')-Y(s))^2]^{\frac{1}{2}}) + K_0(k[(X(s')-X(s))^2+(Y(s')+Y(s))^2]^{\frac{1}{2}})\}\,\mathrm{d}s = -\frac{\partial}{\partial n(s')}K_0(k[(X(s')-\sigma)^2+Y^2(s')]^{\frac{1}{2}}),$$

where $\sigma \in (-\infty, +\infty)$ is a parameter.

To reduce the problem (6), (11)–(13) to the spectral problem for a quadratic operator family we seek u in the form

$$u(x,y) = (V\mu)(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \mu(\sigma)\,g(x,y;\sigma,0)\,\mathrm{d}\sigma, \tag{17}$$

where $\mu \in L_2(-\infty,+\infty)$. The single-layer Green potential (17) clearly satisfies the equation (11), boundary condition (12) and the condition at infinity (6). Moreover we have from (14) and (16)

$$\frac{\partial V\mu}{\partial y} = \mu \quad \text{when} \quad y = 0. \tag{18}$$

Substituting the potential (17) into the condition (13), we get in view of (18)

$$\mu = \frac{(1+\epsilon)\nu}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}x\xi}\frac{\lambda(\lambda\tanh\lambda-\nu)}{(\nu^2+\epsilon\lambda^2)\tanh\lambda-(1+\epsilon)\nu\lambda}(\widetilde{V\mu})\,\mathrm{d}\xi \quad \text{when} \quad y=0. \tag{19}$$

Here

$$\left.\begin{aligned} (V\mu)(x,0) &= \pi^{-1}\int_{-\infty}^{+\infty}\mu(\sigma)\{K_0(k|x-\sigma|)+M(x,\sigma)\}\,\mathrm{d}\sigma,\\ M(x,\sigma) &= 2\int_S m(s,\sigma)\,K_0(k[(x-X(s))^2+Y^2(s)]^{\frac{1}{2}})\,\mathrm{d}s. \end{aligned}\right\} \tag{20}$$

Application of the Fourier transform to (19) gives

$$\tilde{\mu}(\xi) = \frac{(1+\epsilon)\nu\lambda(\lambda\tanh\lambda-\nu)}{(\nu^2+\epsilon\lambda^2)\tanh\lambda-(1+\epsilon)\nu\lambda}(\widetilde{V\mu})(\xi,0).$$

This relation can be written in the form of the square operator pencil with respect to the spectral parameter ν. We have

$$\nu^2[\lambda^{-2}\tilde{\mu}+(1+\epsilon)(\lambda^{-1}\coth\lambda)(\widetilde{V\mu})]-\nu(1+\epsilon)[(\lambda^{-1}\coth\lambda)\tilde{\mu}+(\widetilde{V\mu})]+\epsilon\tilde{\mu}=0.$$

If we apply the inverse Fourier transform, then we arrive at the square pencil

$$\nu^2\left[(2k)^{-1}\int_{-\infty}^{+\infty}\exp(-k|x-\sigma|)\,\mu(\sigma)\,d\sigma+(1+\epsilon)\int_{-\infty}^{+\infty}\mathcal{L}(k,k|x-\sigma|)(V\mu)(\sigma,0)\,d\sigma\right]$$
$$-\nu(1+\epsilon)\left[\int_{-\infty}^{+\infty}\mathcal{L}(k,k|x-\sigma|)\,\mu(\sigma)\,d\sigma+(V\mu)(x,0)\right]+\epsilon\mu(x)=0. \quad (21)$$

Here we used the relation between the convolution operator and the Fourier transform (see, e.g. Vladimirov 1971) and the well-known formula (see e.g. Gradshtein & Ryzhik 1965, 3.723.2)

$$\int_0^\infty \frac{\cos x\xi}{k^2+\xi^2}\,d\xi=\frac{\pi}{2k}e^{-k|x|}.$$

Also, we have introduced the kernel

$$\mathcal{L}(k,k|x|)=\frac{1}{\pi}\int_0^\infty\frac{\cos x\xi}{(k^2+\xi^2)^{\frac{1}{2}}}\coth(k^2+\xi^2)^{\frac{1}{2}}\,d\xi.$$

Below, we shall employ the following notations for the operators involved in (21)

$$A=LV,\quad A_0=A+C,\quad B=L+V,$$

where

$$L\mu=(L\mu)(x)=\int_{-\infty}^{+\infty}\mathcal{L}(k,k|x-\sigma|)\,\mu(\sigma)\,d\sigma,$$

$$V\mu=(V\mu)(x,0)\quad \text{(see (20))},$$

$$C\mu=(C\mu)(x)=(2k)^{-1}\int_{-\infty}^{+\infty}\exp(-k|x-\sigma|)\,\mu(\sigma)\,d\sigma.$$

We consider all these operators in the space $L_2(-\infty,+\infty)$. Their properties are described in the Appendix.

The spectral problem for the square operator pencil (21) is equivalent to the boundary-value problem (11)–(13), (6) which contains the spectral parameter in the boundary condition (13) on the interface.

5. Perturbation method for the eigenvalue problem

In the previous section we obtained the spectral problem for the square operator pencil in $L_2(-\infty,+\infty)$

$$\nu^2(A_0+\epsilon A)\mu-\nu(1+\epsilon)B\mu+\epsilon\mu=0. \quad (22)$$

Since there is the small parameter ϵ on the left-hand side, then this operator pencil will be called the perturbed pencil. The unperturbed one is

$$\nu^2A_0-\nu B=\nu(\nu A_0-B). \quad (23)$$

This pencil is degenerate. Obviously, it has an eigenvalue $\nu=0$. Other eigenvalues should be determined from

$$\nu A_0\mu-B\mu=0. \quad (24)$$

The spectral problem (24) is another form of the usual problem of trapped modes in a channel with homogeneous fluid of density ρ. The fluid has infinite depth, the mean level of the free surface coincides with $y = 1$, and the cylinder is submerged to a depth which is strictly greater than one.

To recognize this character of (24) one has to return to the boundary-value problem (4)–(6). If $\epsilon = 0$, then from (5) we get

$$u_y^* = u_y \quad \text{and} \quad u^* = u \quad \text{when} \quad y = 0.$$

Hence u^* is the unique continuation of u to the strip W^*. After such a continuation the function u is a solution of the following boundary value problem:

$$u_{xx} + u_{yy} = k^2 u \quad \text{in} \quad W^* \cup (\bar{W} \backslash \bar{D}), \tag{25}$$

$$u_y - \nu u = 0 \quad \text{when} \quad y = 1, \tag{26}$$

$$\partial u / \partial n = 0 \quad \text{on} \quad S, \tag{27}$$

$$\int_{-\infty}^{+\infty} u^2(x, 1)\,\mathrm{d}x + \int_{W^* \cup (\bar{W} \backslash \bar{D})} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{28}$$

Thus, the spectral problem (24) is equivalent to (25)–(28) with the spectral parameter in (26). This problem was investigated in detail by Ursell (1987). He reduced (25)–(28) to the spectral problem for a bounded, symmetric, integral operator on the free surface and proved that there are always a finite number of positive point eigenvalues below a certain cutoff frequency. Hence, the problem (24) has the same spectral properties. However, (24) differs in form from the spectral problem that arises in Ursell's (1987) paper, because our integral operators are defined on the immersed horizontal line instead of the free surface.

Since there is the small parameter ϵ in (22), it is convenient to apply the usual perturbation technique. Let us represent the eigenvalue and the eigenfunction in the form (see e.g. Friedrichs 1965).

$$\left.\begin{aligned} \nu &= \nu_0 + \epsilon\nu_1 + \epsilon^2\nu_2 + \dots, \\ \mu &= \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \dots. \end{aligned}\right\} \tag{29}$$

Substituting these expansions into (22) and equating the expressions multiplied by ϵ^0, $\epsilon^1, \epsilon^2, \dots$ respectively, we obtain the system

$$\nu_0(\nu_0 A_0 \mu_0 - B\mu_0) = 0, \tag{30}$$

$$\nu_0(\nu_0 A_0 \mu_1 - B\mu_1) = (\nu_1 + \nu_0) B\mu_0 - \mu_0 - \nu_0(2\nu_1 A_0 \mu_0 + \nu_0 A\mu_0), \tag{31}$$

$$\begin{aligned} \nu_0(\nu_0 A_0 \mu_2 - B\mu_2) = {} & (\nu_2 + \nu_1) B\mu_0 + (\nu_1 + \nu_0) B\mu_1 - \mu_1 - \nu_1^2 A_0 \mu_0 \\ & - 2\nu_0(\nu_2 A_0 \mu_0 + \nu_1 A\mu_0) - \nu_0(2\nu_1 A_0 \mu_1 + \nu_0 A\mu_1) \end{aligned} \tag{32}$$

$$\vdots$$

Equation (30) contains the unperturbed operator pencil (23), which has a zero eigenvalue and the finite set of positive eigenvalues satisfying (24) as was pointed out above.

Let us first construct the expansions (29), starting from the non-degenerate positive eigenvalue $\nu_0^{(+)}$, and from the corresponding eigenfunction of (24) $\mu_0^{(+)}$. The eigenvalue is non-degenerate if the equation

$$\nu_0^{(+)} A_0 \mu - B\mu = f$$

has a solution for every right-hand side f, which is orthogonal to $\mu_0^{(+)}$

$$(f, \mu_0^{(+)}) = \int_{-\infty}^{+\infty} f(x)\,\mu_0^{(+)}(x)\,\mathrm{d}x = 0.$$

We assume that $\mu_0^{(+)}$ is normalized:

$$\|\mu_0^{(+)}\|^2 = \int_{-\infty}^{+\infty} [\mu_0^{(+)}(x)]^2\,\mathrm{d}x = 1.$$

It is clear that the perturbed eigenfunction $\mu^{(+)}$ cannot be found from (30)–(32), ... uniquely, since we can multiply each equation by a different constant. As usual (see e.g. Friedrichs 1965) we introduce the linear condition

$$(\mu^{(+)}, \mu_0^{(+)}) = \int_{-\infty}^{+\infty} \mu^{(+)}(x)\,\mu_0^{(+)}(x)\,\mathrm{d}x = 1,$$

which implies

$$(\mu_1^{(+)}, \mu_0^{(+)}) = (\mu_2^{(+)}, \mu_0^{(+)}) = \ldots = 0.$$

Now, the system (30)–(32), ... can be successively solved with respect to $\mu_1^{(+)}, \mu_2^{(+)}; \ldots$. Equation (31) is solvable if its right-hand side is orthogonal to $\mu_0^{(+)}$. This gives the value

$$\nu_1^{(+)} = \frac{[\nu_0^{(+)}]^2\,(\mu_0^{(+)}, C\mu_0^{(+)}) - 1}{(\mu_0^{(+)}, B\mu_0^{(+)})},$$

where $(\mu_0^{(+)}, B\mu_0^{(+)}) > 0$, since B is a positive operator (see the Appendix). Here (30) and the definition of A_0 should be taken into account. When $\mu_1^{(+)}$ is obtained, $\nu_2^{(+)}$ and $\mu_2^{(+)}$ can be determined in the same way, and so on.

Thus, the eigenvalue $\nu^{(+)}$ of the perturbed problem close to the positive eigenvalue $\nu_0^{(+)}$ of the unperturbed problem can be found to any necessary accuracy. It is clear that $\nu^{(+)} > 0$ if ϵ is sufficiently small, and there is a finite set $\{\nu^{(+)}\}$ of such positive eigenvalues.

Now, let us construct the expansions (29), starting from $\nu_0^{(0)} = 0$, which is an eigenvalue of (30). Then, the system (31)–(32), ... can be rewritten as follows:

$$\nu_1^{(0)}\,B\mu_0^{(0)} - \mu_0^{(0)} = 0, \tag{33}$$

$$\nu_1^{(0)}\,B\mu_1^{(0)} - \mu_1^{(0)} = [\nu_1^{(0)}]^2\,A_0\,\mu_0^{(0)} - \mu_0^{(0)} - \nu_2^{(0)}\,B\mu_0^{(0)}, \tag{34}$$

$$\vdots$$

In (34) equation (33) has been taken into account.

According to (33), we have the usual eigenvalue problem of finding a pair $(\nu_1^{(0)}, \mu_0^{(0)})$. It has a finite set of positive eigenvalues if k is large enough (see the Appendix). Let us choose one non-degenerate eigenvalue $\nu_1^{(0)}$, and let $\mu_0^{(0)}$ be the corresponding normalized eigenfunction. As above, we require

$$(\mu^{(0)}, \mu_0^{(0)}) = 1,$$

which yields

$$(\mu_1^{(0)}, \mu_0^{(0)}) = (\mu_2^{(0)}, \mu_0^{(0)}) = \ldots = 0.$$

This allows the system (34), ... to be solved successively with respect to $\mu_1^{(0)}, \mu_2^{(0)}, \ldots$.

Equation (34) is solvable, if its right-hand side is orthogonal to $\mu_0^{(0)}$. This gives the value

$$\nu_2^{(0)} = \frac{[\nu_1^{(0)}]^2\,(\mu_0^{(0)}, A_0\,\mu_0^{(0)}) - 1}{(\mu_0^{(0)}, B\mu_0^{(0)})},$$

where $(\mu_0^{(0)}, B\mu_0^{(0)}) > 0$, because B is a positive operator (see the Appendix).

Since $\nu_1^{(0)} > 0$, then

$$\nu^{(0)} = \epsilon\nu_1^{(0)} + \epsilon^2\nu_2^{(0)} + \ldots > 0 \tag{35}$$

if ϵ is small enough.

Thus, the following conclusions can be drawn.

(i) Under the assumptions that k is large enough and ϵ is small the perturbed square pencil (22) has two finite sets of positive eigenvalues $\{\nu^{(+)}\}$ and $\{\nu^{(0)}\}$.

(ii) If eigenvalues $\{\nu_0^{(+)}\}$ of (24) are non-degenerate ($\{\nu_0^{(+)}\}$ is a finite set), then each of them induces a positive eigenvalue $\nu^{(+)}$ of the form (29).

(iii) If characteristic values $\{\nu_1^{(0)}\}$ of (33) are non-degenerate ($\{\nu_1^{(0)}\}$ is a finite set if k is large enough), then each of them induces a positive eigenvalue $\nu^{(0)}$ of the form (35) for sufficiently small ϵ.

6. Discussion

What is the hydrodynamical meaning of the eigenvalues obtained in the previous section? Since the eigenvalue $\nu^{(+)}$ is close to the eigenvalue $\nu_0^{(+)}$ and the latter corresponds to a trapped mode on the surface of homogeneous fluid without an interface, then $\nu^{(+)}$ corresponds to a trapped mode of waves on the free surface of two-layer fluid. The eigenvalue $\nu^{(0)}$ is proportional to ϵ and there are no such eigenvalues for the homogeneous fluid. Consequently, this eigenvalue corresponds to a trapped mode of internal waves on the interface between two layers.

It is interesting to estimate the frequency of trapped mode of internal waves. According to the Theorem from the Appendix, $\nu_1^{(0)}$ is close to the characteristic value of the operator $2V$ if k is large enough. It was shown in Ursell (1987) that characteristic values of V correspond to trapped modes of waves on the free surface of the lower fluid in absence of the upper fluid. If we denote by ω_i the frequency of a trapped mode of internal wave and by ω_s the frequency of a trapped mode of surface waves for the lower fluid in the absence of the upper fluid, then we have approximate equality

$$2\omega_i^2 \approx \omega_s^2\epsilon \quad \text{or} \quad \omega_i/\omega_s \approx (\epsilon/2)^{\frac{1}{2}},$$

which is valid if k is large enough.

For three values of the parameter ϵ, which describes some real interfaces between fresh and salt water one can find the following corresponding approximate quotients ω_i/ω_s:

ϵ	0.04	0.02	0.01
ω_i/ω_s	0.14	0.10	0.07.

This decrease in the frequency of trapped internal waves is similar to the decrease in the velocity of internal waves compared with the velocity of surface waves described in Lamb (1932, Art. 231). In the last case the velocity ratio is exactly equal to $[\epsilon/(2+\epsilon)]^{\frac{1}{2}}$.

The method developed above in the case when the lower fluid has infinite depth, is valid without any changes for a two-layer fluid with both layers of finite depth. One has only to take another Green function at the second stage of reduction. Such a Green function for the fluid of finite depth was constructed by Ursell (1987).

The case when the cylinder is immersed totally in the upper layer is more complicated. This problem can be easily reduced to the boundary-value problem which involves only the potential u^* for the upper layer. However, apart from the usual condition on the free surface

$$u_y^* - \nu u^* = 0 \quad \text{when} \quad y = 1, \tag{36}$$

the spectral parameter ν will be included in a pseudo-differential operator in the boundary condition on the interface. Then we have to use another Green's function satisfying (36). Such a kind of Green function was also constructed by Ursell (1987), but this Green function depends on ν. Hence, one obtains a spectral problem with an operator function depending on ν in a way which is not so simple as square operator pencil.

The author thanks Dr J. Grue for the possibility to visit the University of Oslo where this work was completed.

Appendix. Properties of the operators from §4

We first consider the properties of C and L, since these operators have explicit kernels.

PROPOSITION 1. *The operators C and L both are symmetric, positive and bounded operators.*

Proof. Due to Parseval's theorem and to the definition of C, we have

$$(C\mu,\mu) = (2\pi)^{-1}(\widetilde{C\mu},\tilde{\mu}) = (2\pi)^{-1}\int_{-\infty}^{+\infty}\frac{\tilde{\mu}^2(\xi)}{k^2+\xi^2}\,d\xi.$$

Now, it is clear that $(C\mu,\mu) > 0$, if $\mu \not\equiv 0$. Then C is a positive, symmetric operator. Furthermore, $(k^2+\xi^2)^{-1} \leqslant k^{-2}$. Hence,

$$(C\mu,\mu) \leqslant k^{-2}\|\mu\|^2,$$

which means that C is a bounded operator.

In the same way

$$(L\mu,\mu) = (2\pi)^{-1}(\widetilde{L\mu},\tilde{\mu}) = (2\pi)^{-1}\int_{-\infty}^{+\infty}\frac{\coth\lambda}{\lambda}\tilde{\mu}^2(\xi)\,d\xi,$$

where $\lambda = (k^2+\xi^2)^{\frac{1}{2}}$. Then L is a positive, symmetric operator.

Since

$$\left(\frac{\coth\lambda}{\lambda}\right)' = \frac{1}{\lambda\sinh^2\lambda}-\frac{\coth\lambda}{\lambda^2} = -\frac{1}{\lambda^2\sinh^2\lambda}\left(\frac{\sinh 2\lambda}{2}-\lambda\right) < 0,$$

then

$$\frac{\coth\lambda}{\lambda} \leqslant \frac{\coth k}{k}$$

and L is a bounded operator, whose norm does not exceed $k^{-1}\coth k$. □

The operator V was introduced and investigated by Ursell (1987). The properties of V can be summarized in the following two propositions.

PROPOSITION 2. *The operator V is a symmetric, positive and bounded operator in $L_2(-\infty,+\infty)$.*

We denote by $\|V\|$ the norm of V in $L_2(-\infty,+\infty)$. If we consider the spectral problem

$$(V-\lambda I)\mu = 0$$

then we have

PROPOSITION 3. *The spectrum of V is real, continuous for* $0 < \lambda < k^{-1}$ *and discrete for* $k^{-1} < \lambda < \|V\|$. *Moreover, there exists at least one point eigenvalue when S is not an empty set.*

From Propositions 1 and 2 it immediately follows that

COROLLARY 1. *Operators* A, A_0 *and* B *are bounded and* B *is symmetric, positive.*

This Corollary follows from the definition of A, A_0 and B.

THEOREM. *If k is large enough, then there exists at least one positive point eigenvalue of the operator B. The number of positive point eigenvalues of B is finite. These eigenvalues are close to the point eigenvalues of the operator 2V.*

Proof. Let us write

$$B = L + V = 2V + (L - V).$$

Since the operators involved are symmetric, positive and bounded, then according to the general perturbation theory (see Kato 1966) it is enough to show that the norm of the operator $L - V$ is small, if k is large enough. Really, from the Proposition 3 it follows that the operator $2V$ has a finite number (at least one) of positive point eigenvalues. Hence, if $L - V$ has a small norm, then the eigenvalues of B exist and are positive and close to the eigenvalues of $2V$.

The kernel of L can be represented in the form

$$Ł(k, k|x|) = \frac{1}{\pi}\int_0^\infty \frac{\coth\lambda}{\lambda}\cos x\xi \, d\xi = \frac{1}{\pi}\int_0^\infty \frac{\cos x\xi}{\lambda} d\xi + \frac{2}{\pi}\int_0^\infty \frac{\cos x\xi}{\lambda}\frac{e^{-2\lambda}}{1-e^{-2\lambda}} d\xi.$$

From Gradshtein & Ryzhik (1965, formula 6.671.14) we have for the Fourier transform of the Macdonald function

$$2\int_0^\infty K_0(kx)\cos x\xi \, dx = \pi/(k^2+\xi^2)^{\frac{1}{2}} = \pi/\lambda.$$

Comparing this formula with the previous one, we find

$$Ł(k, k|x|) = \pi^{-1}K_0(k|x|) + N(k, k|x|),$$

where

$$N(k, k|x|) = \frac{2}{\pi}\int_0^\infty \frac{\cos x\xi}{\lambda}\frac{e^{-2\lambda}}{1-e^{-2\lambda}} d\xi.$$

Thus, the operator $L - V$ has the kernel

$$N(k, k|x-\sigma|) - \pi^{-1}M(x, \sigma).$$

Here we also used (20). We shall demonstrate that each of the operators with kernels N and M is small if k is large enough.

Since the kernel N depends on $|x-\sigma|$, it is convenient to apply Parseval's theorem. Then, we have

$$\int_{-\infty}^{+\infty} \mu(x)\,dx \int_{-\infty}^{+\infty} N(k, k|x-\sigma|)\mu(\sigma)\,d\sigma = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1}{\lambda}\frac{e^{-2\lambda}}{1-e^{-2\lambda}}\tilde{\mu}^2(\xi)\,d\xi.$$

Considering the function

$$f(x) = \frac{e^{-x}}{x(1-e^{-x})}$$

we conclude that it is monotonically decreasing for $x > 0$, since

$$f'(x) = \frac{-xe^{-x}(1-e^{-x})-e^{-x}[(1-e^{-x})+xe^{-x}]}{x^2(1-e^{-x})^2} < 0.$$

Hence, the norm of the operator with the kernel $N(k, k|x-\sigma|)$ does not exceed the value

$$\frac{2}{k}\frac{e^{-2k}}{1-e^{-2k}},$$

which is small if k is large enough.

Now, let us estimate the norm of the operator with the kernel $M(x, \sigma)$. Due to the Schwarz inequality we can write

$$\int_{-\infty}^{+\infty} dx \left(\int_{-\infty}^{+\infty} M(x,\sigma)\,\mu(\sigma)\,d\sigma\right)^2 \leqslant \|\mu\|^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M^2(x,\sigma)\,dx\,d\sigma.$$

Then according to (20) and the Schwarz inequality

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M^2(x,\sigma)\,dx\,d\sigma$$

$$\leqslant 4\left\{\int_{-\infty}^{+\infty} d\sigma \int_S m^2(s,\sigma)\,ds\right\}\left\{\int_{-\infty}^{+\infty} dx \int_S [K_0(k\{(x-X(s))^2+Y^2(s)\}^{\frac{1}{2}})]^2\,ds\right\}.$$

Hence the norm of the operator with the kernel $M(x, \sigma)$ is small for large enough values of k, because of the asymptotic behaviour of the Macdonald function at infinity. Here we used the fact that the distance between S and the interface is positive. □

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